Pattern interaction and spiral waves in a two-layer system of excitable units

V. I. Nekorkin and V. B. Kazantsev
Radiophysical Department, Nizhny–Novgorod State University, 23, Gagarin Avenue, 603600 Nizhny–Novgorod, Russia

M. G.Velarde*
Instituto Pluridisciplinar, Universidad Complutense de Madrid, Paseo Juan XXIII, 1, 28040 Madrid, Spain

L. O. Chua
Department of Electrical Engineering and Computer Science, University of California at Berkeley, Berkeley, CA 94720
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A system composed of two coupled lattices, hence a layered structure, is studied when the unit at each site is an *active* electronic circuit possessing two accessible stable steady states. In the absence of interlattice coupling, each lattice taken separately represents a discrete, reaction diffusion system. We show that, depending on the strength of the diffusion coefficient, each lattice may exhibit either a wide variety of stable steady patterns or a number of different wave patterns including rotating spirals. Moreover, for fixed reaction kinetics each lattice can exhibit spiral waves of both *excitable* and *oscillatory* type. For nonoscillating kinetics, the metastable periodiclike behavior of the unit is at the origin of the oscillatory spirals. From initially different global patterns or waves in each lattice, the interaction may lead to synchronization and hence a new (controlled) form and the replication of a given one. We also show how there is reentry of spiral waves between the two coupled layers associated with the ''competition'' of their *oscillatory* and *excitable* spiral wave properties. [S1063-651X(98)08808-4]

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I. INTRODUCTION

Interaction of patterns (steady and traveling wave patterns) is an important, widespread problem in various physical, chemical, and biological systems. Examples are the interaction of vortices in a system of two coupled fibers [1–3], fluxons in a system of coupled Josephson junctions [4], the competition between stationary Turing patterns with traveling waves in isothermal single-phase chemical systems [5], the interaction of spiral waves in two-layer cellular neural networks [6], etc. In spite of their different physical origins, the process of their interaction should have common, universal features. This can be established by using two coupled lattices or subsystems, which is also a three-dimensional layered structure. Thus, in particular, we consider a system of two interacting, two-dimensional lattices of resistively coupled electronic units, each of which is known to provide an *active* discrete medium for generating stationary patterns [7,8], various types of nonlinear waves including wave fronts [7,9], pulses and pulse trains [3,10], target waves and rotating spirals [11,12], spatial disorder [7] and spatiotemporal chaos [13]. The resistive (diffusive) connection between the units allows one to treat such a lattice as a discrete analog of a multispecies reaction diffusion system. The ''reaction kinetics'' in this case is defined by the dynamics of the single electronic unit. We choose here a unit described by a third-order system with a number of parameters allowing a rich variety of dynamic behaviors from simple stable steady states to complex chaotic oscillations [14,15]. By varying the ''kinetics,'' the lattice of electronic units can be considered as a qualitative model to simulate various features of an active medium [16–22].

The paper is organized as follows. In Sec. II, we introduce the model problem, and give details of a unit or element in a lattice site. In Sec. III we describe the dynamics of a single lattice and its units for a given reaction kinetics. We prove its multistability, allowing a wealth of stable steady patterns. Changing the coefficient of diffusion between the intralattice units, we find that multistability of different wave patterns is also possible, including the coexistence of ''excitable'' and ''oscillatory'' spiral waves that are actually realizable according to the specific choice of initial conditions (i.c.). In Secs. IV and V, we study a two-layer structure with two interacting lattices. We prove that for certain conditions there may be synchronization of all kinds of global motions in the layers. We provide a few illustrative examples of this synchronization, demonstrating how pattern interaction occurs between the layers and how different types of rotating spiral waves compete with each other.

II. MODEL

In view of what follows let us consider a system of two coupled identical lattices where the interlattice interaction between nearest neighbors is a ''point by point'' resistive coupling h, as shown in Fig. 1. Each site is occupied by an electronic circuit, the unit or element. The number of such units is $N\times N$, and they are also resistively coupled between nearest neighbors (intralattice coupling coefficient D). The dynamics of these two coupled lattices is described by the following equations:

*Author to whom correspondence should be addressed.

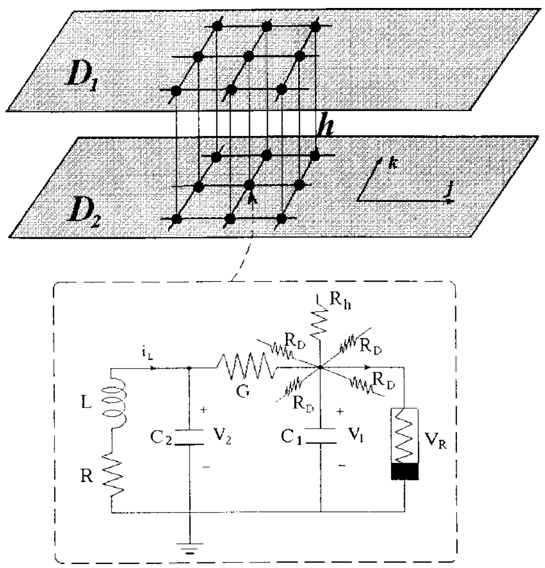

FIG. 1. Schematic diagram of the two-layer structure and of the electronic unit here used as an *active* element in each lattice.

$$\dot{x}^1_{j,k}=\alpha[y^1_{j,k}-x^1_{j,k}-f(x^1_{j,k})]+D_1(\Delta x^1)_{j,k}+h(x^2_{j,k}-x^1_{j,k}),$$
$$\dot{y}^1_{j,k}=x^1_{j,k}-y^1_{j,k}+z^1_{j,k},$$
$$\dot{z}^1_{j,k}=-\beta y^1_{j,k}-\gamma z^1_{j,k},$$
$$\dot{x}^2_{j,k}=\alpha[y^2_{j,k}-x^2_{j,k}-f(x^2_{j,k})]+D_2(\Delta x^2)_{j,k}+h(x^1_{j,k}-x^2_{j,k}), \tag{1}$$
$$\dot{y}^2_{j,k}=x^2_{j,k}-y^2_{j,k}+z^2_{j,k},$$
$$\dot{z}^2_{j,k}=-\beta y^2_{j,k}-\gamma z^2_{j,k},$$
$$j,k=1,2,\ldots,N,$$

where the superscripts 1 and 2 denote the variables of the first and the second lattice, respectively; $(\Delta w)_{j,k}$ is the discrete Laplace operator

$$(\Delta w)_{j,k}=w_{j+1,k}+w_{j-1,k}+w_{j,k+1}+w_{j,k-1}-4w_{j,k};$$

D_1 and D_2 are generally different; h characterizes the interlattice or interlayer coupling which we also consider diffusive; and $f(x)$ is the three-segment piecewise-linear function [14]

$$f(x)=\begin{cases} b_1x-a-b_1 & \text{if } x\geqslant 1\\ -ax & \text{if } -1<x<1\\ b_2x+a+b_2 & \text{if } x\leqslant -1\,, \end{cases} \tag{2}$$

with $a>0$ and $b_1,b_2>0$. The other parameters of the system, α, β, and γ, are also taken positive. We assume zero-flux (Neumann) boundary conditions for each layer.

We fix the parameters of the unit, hence we fix the reaction kinetics of our discrete resistive-diffusive system to have *bistable* and *excitable* properties. Changing D allows us to proceed from a dynamical system to a spatially extended one and its space-time dynamics. Then in the lattice we choose either pattern formation (no waves) or (excitable) wave behavior. For the purpose of our research here, we take

$$\alpha=2.5,\quad \beta=0.5,\quad \gamma=0.01,\quad a=1.5,\quad b_1=b_2=b=2. \tag{3}$$

This choice provides an interesting behavior of the unit or reaction kinetics, as we shall see later on. Consequently, in what follows we restrict consideration to the case of varying diffusion coefficient and interlayer coupling only. Thus we first study the dynamical system corresponding to the *active* unit before proceeding to the space-time dynamics of one and two coupled lattices. Then there are three steady states (fixed points) in the phase space of the unit:

$$O(0,0,0),\quad P^+(x_0,y_0,z_0),\quad P^-(-x_0,-y_0,-z_0),$$

with

$$x_0=\frac{(b+a)(\gamma+\beta)}{[\gamma b+\beta(b+1)]},\quad y_0=\frac{(b+a)\gamma}{[\gamma b+\beta(b+1)]},$$
$$z_0=-\frac{(b+a)\beta}{[\gamma b+\beta(b+1)]}.$$

For the chosen parameter values, the ''outer'' points P^+ and P^- are stable foci, while the origin O is a saddle. Thus we have bistable reaction kinetics. By varying the intralattice diffusion coefficients D_1 and D_2, we show in Sec. III that for such a unit a single lattice (a layer) represents a discrete medium capable of exhibiting stable steady spatial structures and a variety of spiral wave patterns.

III. PATTERNS AND WAVES IN A SINGLE LATTICE

Let us consider the dynamics of a single lattice $(h=0)$. It is given by the system:

$$\dot{x}_{j,k}=\alpha[y_{j,k}-x_{j,k}-f(x_{j,k})]+D(\Delta x)_{j,k},$$
$$\dot{y}_{j,k}=x_{j,k}-y_{j,k}+z_{j,k}, \tag{4}$$
$$\dot{z}_{j,k}=-\beta y_{j,k}-\gamma z_{j,k},$$
$$j,k=1,2,\ldots,N.$$

A. Pattern formation: regular and chaotic patterns

To show the possibility of pattern formation in the lattice, we note that system (4) has 2^{N^2} stable steady states when its parameters are located in the domain $D_{\rm ch}$ defined by the inequality [23]

$$D<D^*=\frac{\alpha[a(\gamma+\beta)-\beta)(\gamma b+\beta(b+1)]}{4(\gamma+\beta)[\beta+(\gamma+\beta)(a+2b)]}. \tag{5}$$

For the parameter set (3), the critical value of the diffusion coefficient is $D^*\approx 0.15$. Each of these states defines a steady pattern in the $\{\mathbb{Z}^2,\mathbb{R}\}$ state space. Moreover, every pattern can be coded by $N \times N$ matrix of two symbols (for example, 0 and 1), and any given $N \times N$ matrix defines a possible spatial configuration of the pattern. Thus there exists a wealth of possible spatial patterns from simple homogeneous, periodic, regular to disordered, or spatially chaotic. Low enough diffusion [Eq. (5)] implies the bistable character in the spatial distribution of the species (state variables) in the lattice. The coordinates $(x^*_{j,k}, y^*_{j,k}, z^*_{j,k})$ of the steady states are located in neighborhoods (''absorbing domains'') of the fixed points $P^+(x_0, y_0, z_0)$ and $P^-(-x_0, -y_0, -z_0)$ of the unit at each site.

Solving the eigenvalue problem for the steady state solution of Eq. (4), we find that for the chosen parameter values [Eq. (5)] each state is locally asymptotically stable. Thus, for weak enough intralattice diffusion D [Eq. (5)], the lattice with the chosen ''bistable kinetics'' exhibits a large number of stable steady patterns. The bistable character of the ''species'' distribution along the lattice makes any two-dimensional black (0) and white (1) picture a steady pattern.

B. Spiral wave patterns

Let us consider how different spiral waves occur in the lattice for our reaction kinetics with high enough intralattice diffusion D, thus leaving the domain of pattern formation [Eq. (5)]. In a similar context but for different kinetics from that chosen here, a number of spiral wave patterns have been observed in a single lattice of the same electronic units [11,12].

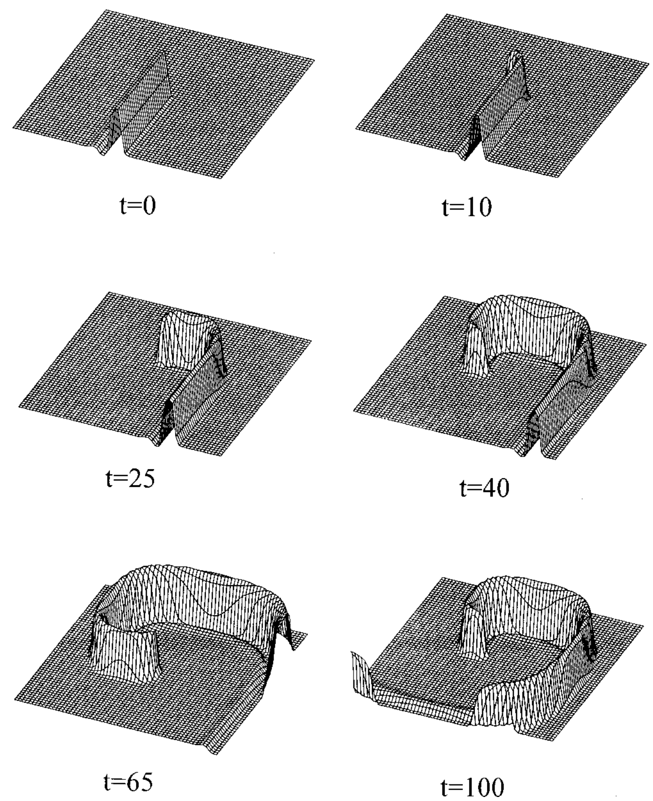

FIG. 2. Evolution of a ''bright'' *excitable* spiral wave following the rupture of the plane wave front. Intralattice diffusion $D=0.8$.

1. ''Dark'' and ''bright'' spiral waves

An array (chain) of electronic units in the described bistable mode supports a variety of stationary pulses and pulse trains including complex or chaotic profiles. Very much like in optical fibers (see, e.g., Ref. [24]) there are ''dark'' and ''bright'' pulses propagating along the ''background'' homogeneous states P^+ and P^-, respectively. The solution $\{x_j(t), y_j(t), z_j(t)\}$ corresponding to a pulse in the chain defines a plane wave solution $\{x_{j,k} \equiv x_j(t), y_{j,k} \equiv y_j(t), z_{j,k} \equiv z_j(t)\}$ in the two-dimensional lattice.

Let us take the plane wave corresponding to a single bright pulse of the one-dimensional chain, and break this front at some instant of time (Fig. 2). Similarly to the waves in excitable media [17,18] the edge of the front starts to twist and after some time forms a stationary spiral wave of bright type (see the sequence of pictures in Fig. 2). Due to the reflection symmetry of the system, there can be dark spirals having originated from dark plane pulses. Figure 3 illustrates a fully developed spiral rotating around its core. As in other excitable media, the core consists of unexcited cells, while the other cells exhibit time-periodic, phase-shifted pulses as shown in Figs. 3(b) and 3(c). Note that the lattice supports the spiral waves only when diffusion D is strong enough. Due to Eq. (5), when decreasing diffusion the spiral wave yields to a steady pattern.

Thus, for the parameter set (3) which provides the bistable mode of the electronic unit and for strong enough intralattice diffusion, the lattice is indeed a discrete reaction diffusion medium with two excitable states. It follows from the symmetry of function $f(x)$ [see Eq. (3)] that the two states have equivalent properties allowing in the lattice dark and bright excitable spirals for the same parameter values. Note that we have considered only the simplest, basic types of spiral waves. More complex wave patterns can also be excited, including those obtained from the multihump one-dimensional pulse trains [10] and ''multiarmed'' spiral wave solutions [17].

2. Oscillatory spiral waves

When a unit exhibits a limit cycle the lattice behaves like an oscillatory medium. Periodic trains in the one-dimensional (1D) case (chain), and target patterns and oscillatory spirals in a 2D system, are examples of the typical processes in such a medium [17]. The spirals in an oscillatory medium look rather different from those found in excitable systems [19]. In an oscillatory medium at each point (here a lattice site), the system is oscillating around the same limit cycle of the reaction kinetics. Diffusion provides a global phase coherence of the local oscillations, hence the spiral wave [16,19]. On purpose, the chosen reaction kinetics [Eqs. (2) and (3)] does not have a limit cycle, as it operates only with two stable fixed points. However, we have found that lattice (4) with Eqs. (2) and (3) supports spiral waves which behave like the waves of an oscillatory medium. Figure 4 illustrates the fully developed oscillatory spiral obtained in the lattice for the same kinetics and diffusion coefficient as for the excitable waves described in Sec. III A. The behavior of the spiral core and phase-shift relations between the other cells are illustrated in Figs. 4(b) and 4(c). A cell near the core [Fig. 4(c)] oscillates (dashed curve) with the same period and slightly smaller amplitude than the other cells (solid curve). This type of spiral wave can be obtained, for example, when

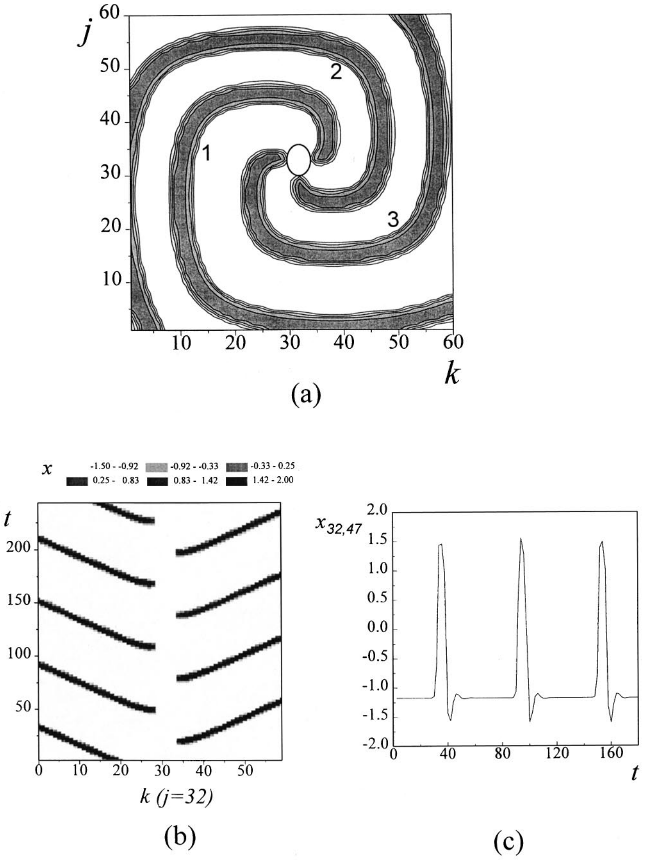

FIG. 3. Fully developed excitable spiral wave for $D=0.8$. (a) 1, 2, and 3—snapshots (x variable) of the wave rotating around its core. (b) Space-time diagram of the section of the lattice along $j=32$. (c) Oscillations of an element far away from the spiral core. Units according to adimensionalization in Eqs. (4).

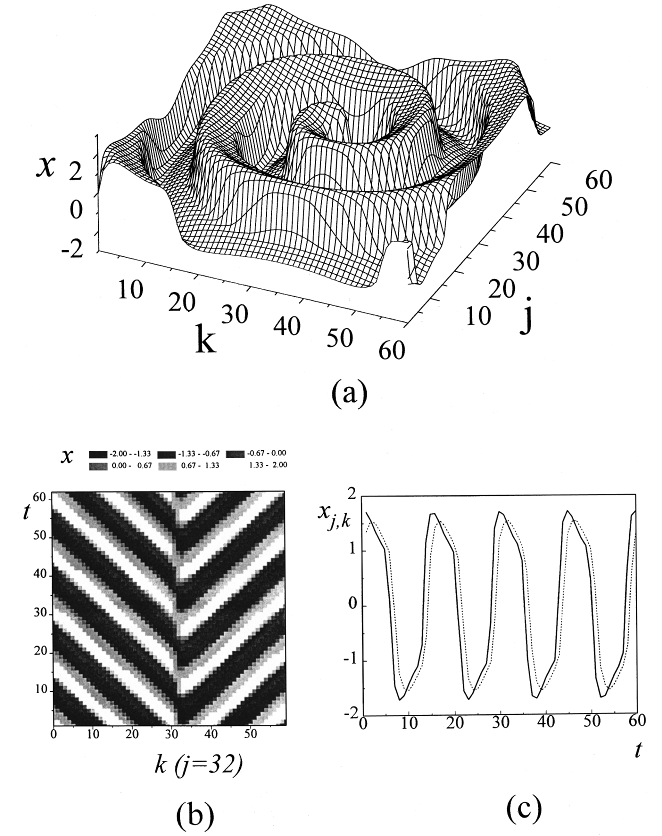

FIG. 4. Fully developed *oscillatory* spiral wave for $D=0.8$. (a) A snapshot (x variable) of the wave. (b) Space-time diagram of the section of the lattice along $j=32$. (c) Oscillations of an element near the spiral core (dashed curve) relative to the oscillations of the other units (solid curve). Units according to adimensionalization in Eqs. (4).

studying wave reentry in the two-layer system, as we show in Sec. IV. What mechanism underlies the oscillatory spiral wave in a medium which has a nonoscillating reaction kinetics?

3. Metastable periodic behavior of the excitable unit as a possible origin of the oscillatory properties of the medium

Let us show that possible oscillatory motions in the third-order phase space of the unit can be described by a 1D Poincaré return map. To derive this map, a procedure similar to that described in Ref. [25] can be used.

Let us choose the plane $x=1$ as a Poincaré surface in the 3D phase space of the unit. Just taking the points of the intersection of a unit trajectory when it leaves the linear region $|x|>1$, we obtain a 2D return map. All these points forming the 2D map are located very near to some line l_S. This line is the intersection of the two-dimensional stable manifold (plane) of the steady state P^+ and the plane $x=1$. In other words, while traveling within the region $x>1$, the trajectories of the unit come very near to the manifold when crossing the secant plane $x=1$. This allows us to characterize possible oscillatory motions of the unit by a 1D Poincaré return map $\overline{S}=h(S)$, where S accounts for the y coordinate of the points of the line l_S. The function $h(S)$ is derived by a numerical integration of the equations describing the unit.

The behavior of the map for different values of b_2, taken as a control parameter from set (3), is shown in Fig. 5(a). A saddle-node bifurcation occurs in the system for a certain value of b_2. The map in case 1 has a stable fixed point, hence the unit exhibits limit cycle behavior. Figure 5(b) shows its time evolution. Then the lattice in this case actually represents an oscillatory medium. Further increasing the values of b_2, the fixed point disappears, and the map is described by curve 3 [Fig. 5(a)]. Then the trajectories of the system can stay for a rather long time near the location of the limit cycle before decaying to one of the two stable steady points [Fig. 5(c)]. Such a long time (metastable) oscillatory behavior is very much like the laminar (periodic) stage in intermittency [26]. The intralattice diffusion acting as a ''rigid'' mechanism profits from such long lasting local oscillations, to provide ''global coherence'' to the units, hence the self-sustained oscillatory spiral wave. How long the metastable oscillations must last (how far can the map stand from the bifurcation point [Fig. 5(a)]) is defined by the ratio between the characteristic time scale of the local oscillations, τ_L [Fig. 5(c)], and that of the diffusion, τ_D. The time τ_D can be estimated, for example, from the velocity c of the plane wave fronts propagating for a given diffusion coefficient D [10]. For the parameters of Fig. 4, we have $\tau_D \sim 1$. Then the relation $\tau_D \ll \tau_L$ allows the diffusion to create the oscillatory spiral wave as a coherent oscillation of the units, without the

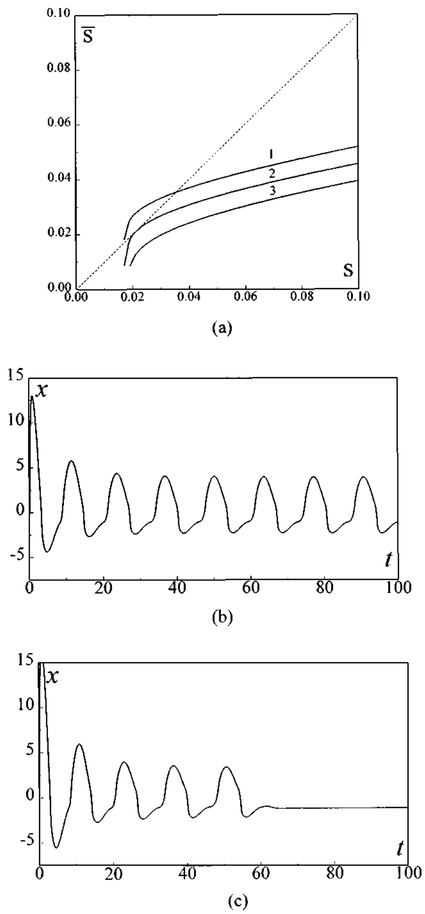

FIG. 5. (a) One-dimensional Poincaré return map characterizing *oscillatory* behavior of the unit near the saddle-node bifurcation. Curve 1: $b_2=0.1$; curve 2: $b_2=0.148$; curve 3: $b_2=0.2$. (b) Time evolution of the limit cycle corresponding to the fixed point of map 1. (c) Typical periodiclike time evolution for map 3 displaying a long lasting, metastable, periodiclike behavior. Units according to adimensionalization in Eqs. (4).

units themselves being in a true limit cycle. A similar argument underlies earlier numerical results reported in Refs. [11,12].

Note, that for our reaction kinetics [Eq. (3)] ($b_2=b=2$), the map stands quite far from the bifurcation point, and hence the oscillations of the unit decay rather quickly. However, since the characteristic period of the periodiclike oscillations is rather long [$T\sim 15$ from Figs. 5(b) and 5(c)], just a few periods are enough for the relation $\tau_D \ll \tau_L$ to be satisfied, and hence to create and sustain an oscillatory spiral wave actually observed in the numerical simulations (Fig. 4).

IV. LAYERED STRUCTURE WITH TWO COUPLED LATTICES

Let us now consider the layered system (1) when the interlayer interaction is switched on ($h>0$). We shall show how, during their time evolution, the (mutual) interaction between two subsystems, one organized in a given form and the other spatially disordered, leads to replication of the form in the initially disordered subsystem. There will be a case when these two competing states are at about the same potential; the initial conditions determine the winner, and among the initial conditions a disordered state is less preferable, hence the loser.

A. Mutual synchronization of motions in the coupled two-layer system

Let us introduce new variables as follows:

$$u_{j,k}=x^1_{j,k}-x^2_{j,k}, \quad v_{j,k}=y^1_{j,k}-y^2_{j,k}, \quad w_{j,k}=z^1_{j,k}-z^2_{j,k},$$

$$u^+_{j,k}=x^1_{j,k}+x^2_{j,k}, \quad v^+_{j,k}=y^1_{j,k}+y^2_{j,k}, \quad w^+_{j,k}=z^1_{j,k}+z^2_{j,k}.$$

Their evolution is described by the system

$$\begin{aligned}
\dot u_{j,k}&=\alpha[v_{j,k}-u_{j,k}-G(u_{j,k},u^+_{j,k})]\\
&\quad+D^+(\Delta u)_{j,k}+D^-(\Delta u^+)_{j,k}-2hu_{j,k},\\
\dot v_{j,k}&=u_{j,k}-v_{j,k}+w_{j,k},\\
\dot w_{j,k}&=-\beta v_{j,k}-\gamma w_{j,k},\\
\dot u^+_{j,k}&=\alpha[v^+_{j,k}-u^+_{j,k}-G^+(u_{j,k},u^+_{j,k})]\\
&\quad+D^+(\Delta u^+)_{j,k}+D^-(\Delta u)_{j,k},\\
\dot v^+_{j,k}&=u^+_{j,k}-v^+_{j,k}+w^+_{j,k},\\
\dot w^+_{j,k}&=-\beta v^+_{j,k}-\gamma w^+_{j,k},\\
&j,k=1,2,\ldots,N,
\end{aligned} \tag{6}$$

with

$$D^-\equiv\frac{D_1-D_2}{2}, \quad D^+\equiv\frac{D_1+D_2}{2},$$

$$G(u_{j,k},u^+_{j,k})=f(x^1_{j,k})-f(x^2_{j,k}),$$

$$G^+(u_{j,k},u^+_{j,k})=f(x^1_{j,k})+f(x^2_{j,k}).$$

1. Identical layers ($D_1=D_2=D$)

It follows from Eqs. (6) that when $D_1=D_2=D$, a synchronization manifold defined by $M=\{u_{j,k}=v_{j,k}=w_{j,k}=0, \quad j,k=1,2,\ldots,N\}$ exists in the phase space of system (6). It can be proved (see the Appendix for details) that for

$$h>\alpha a \tag{7}$$

the manifold M is *globally* asymptotically stable. Therefore, any initial conditions in the identical coupled layers tend to the manifold M, where the synchronized motions are governed by system (4), describing the dynamics of a single lattice.

2. Layers with different diffusion coefficients $D_1\neq D_2$

Consider system (6), with a small parameter $\mu=1/h\ll 1$ representing the coefficient of the derivative $\dot u_{j,k}$. In this case the motions of Eqs. (6) have both fast and slow features [27]. In the phase space there exists a stable surface of slow motions, and all trajectories of Eqs. (6) after some time be-

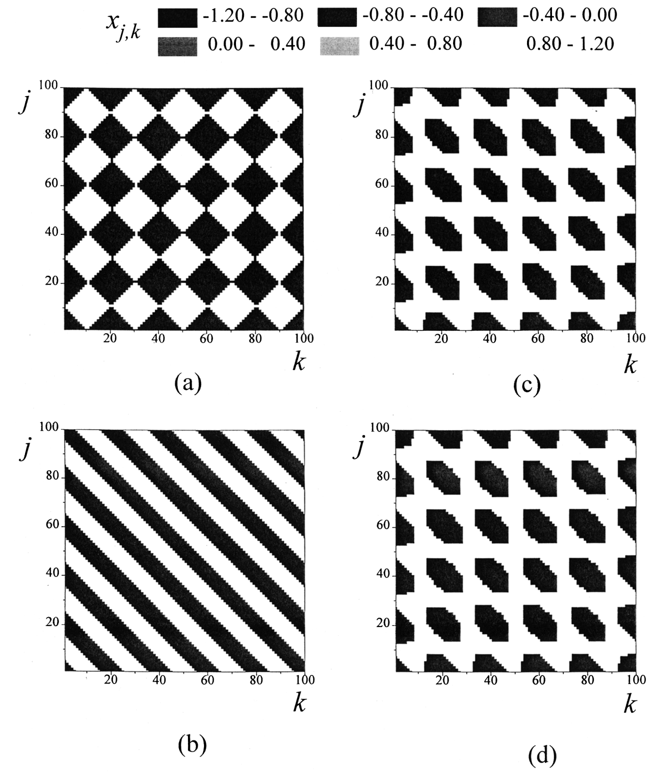

FIG. 6. Mutual synchronization of two steady patterns of regular spatial profiles. Parameter values: $D_1=D_2=0.15$ and $h=3.75$. (a) and (b) Initial patterns. (c) and (d) Synchronized patterns displaying a new common spatial form.

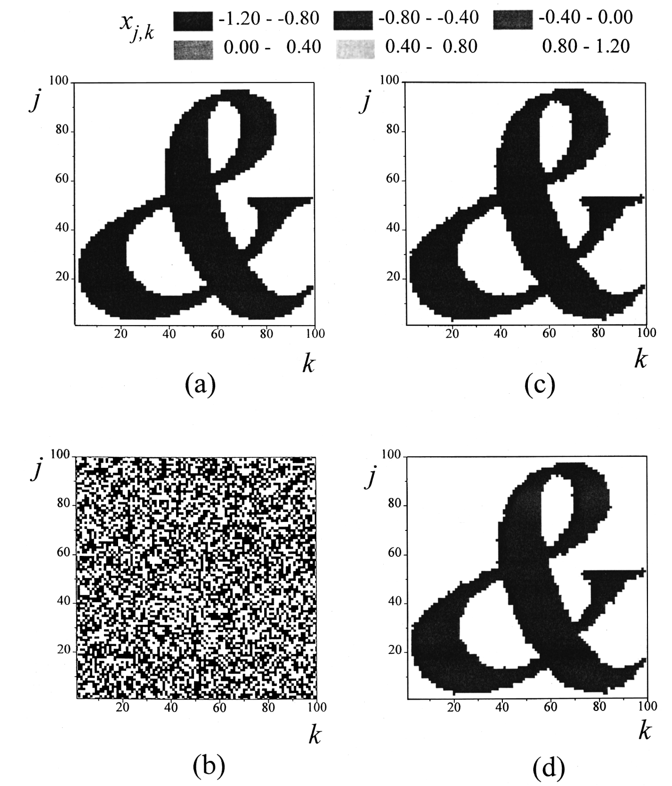

FIG. 7. Replication of form when two lattices are coupled. Parameter values: $D_1=D_2=0.14$ and $h=3.75$. (a) Patterned stimulus. (b) Disordered pattern. (c) and (d) Synchronized state. Both layers exhibit a common and rather faithful copy of the stimulus.

come restricted within thin layers (whose thickness is of order of μ) near this surface. This coincides with the synchronization manifold M. Then when $\mu=0$ the approximate system of slow motions is given by system (4) with the intralattice diffusion $D^+=(D_1+D_2)/2$. Hence for $h\to\infty$ the motions occur in two stages. Any initial conditions in the first (fast) stage rapidly come to the stable surface of slow motions (synchronization manifold M). In the second (slow) stage, motions are governed by system (4), defining the dynamics of the single lattice with $D=D^+$, and tend to an attractor of this system (a pattern or a spiral wave).

B. Pattern synchronization and replication of form

When the diffusion coefficients $D_1=D_2=D$ belong to domain (5), condition (7) ensures the synchronization of steady patterns in system (1). The two layers initially containing different steady patterns evolve to a state with a common steady pattern. Figure 6 illustrates the result of the synchronization of two different regular patterns [Figs. 6(a) and 6(b)]. It is also a regular, steady pattern [Figs. 6(c) and 6(d)], identical for both layers. Note that the terminal state of each lattice exhibits a new spatial form quite different from the initial patterns. This is a new quality born from the autonomous evolution of the corresponding dynamical system. Hence the synchronization process can be considered as a form of global *self-organization* in the three-dimensional structure of two coupled lattices.

When a single lattice is in a steady, spatially disordered pattern, it can be treated as a kind of ''raw material.'' Let us assume that one layer is in such a disordered pattern, while the second layer carries an ''encoded'' structure as a stable steady pattern, given as a stimulus. As the result of the interlattice interaction, there is a *replication of form*. It appears as if such a disordered state is enslaved by the form of the patterned stimulus. The rationale behind this result is the following. Inequalities (5) and (7) demand that the intralattice coupling D be weak enough, while the interlattice coupling h is strong enough. Because each lattice element is *bistable*, the overall interaction can be considered as a competition between two possible locally stable states. Then the small, but non-negligible, value of D provides the initial condition for this competition. For instance, the elements taken from the pattern carrying a regular form may have predominance over those of the disordered pattern. Accordingly, both competing states are at the same potential; the initial conditions determine the winner, and among the initial conditions a disordered state is less preferable, hence the loser. Figure 7 illustrates this process in the two-layer lattice. Figures 7(a) and 7(b) show initial states of the layers. The self-replicated, synchronized patterns are shown in Figs. 7(c) and 7(d). They are quite faithful copies of the initial regular form. The replication process can be considered as a kind of *competition* between the initial patterns [28,29].

C. Controlled spiral waves in the two-layer system

Let us consider system (1), when each layer taken independently can support different spiral wave patterns (see Sec. III B). The investigation of interactions and control of rotat-

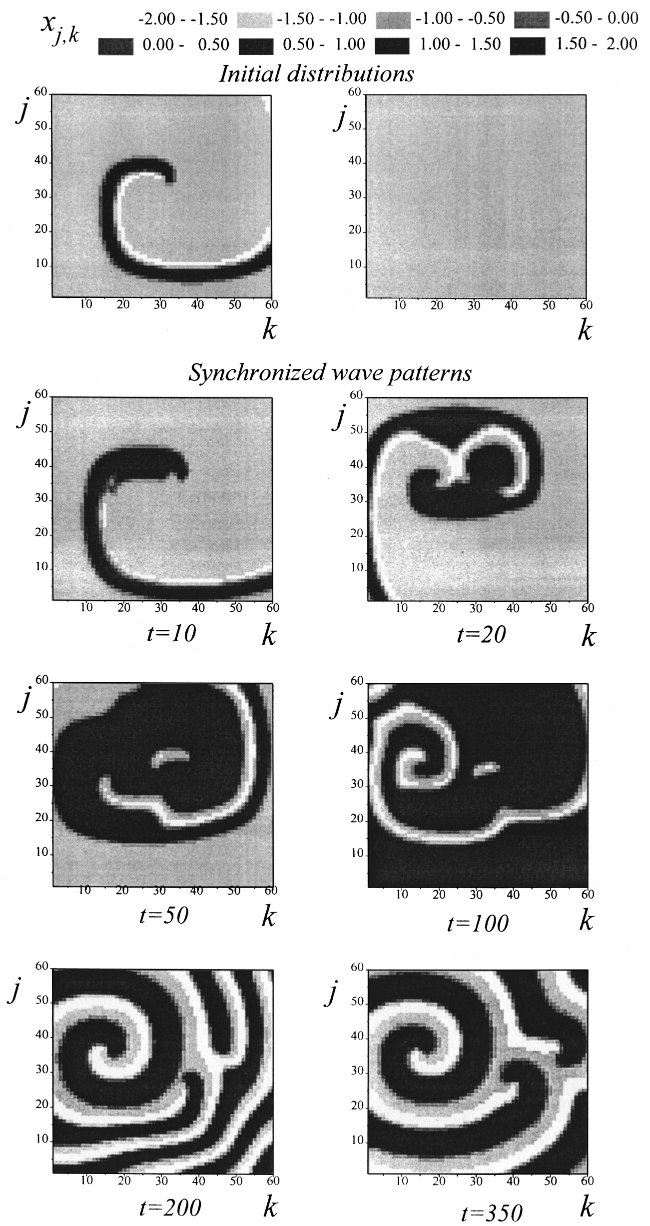

FIG. 8. Time sequence illustrating the appearance of an *oscillatory* spiral wave pattern as the result of the mutual interaction of an *excitable* spiral wave with the unexcited homogeneous state. Parameter values: $D_1 = D_2 = 0.8$ and $h = 1$.

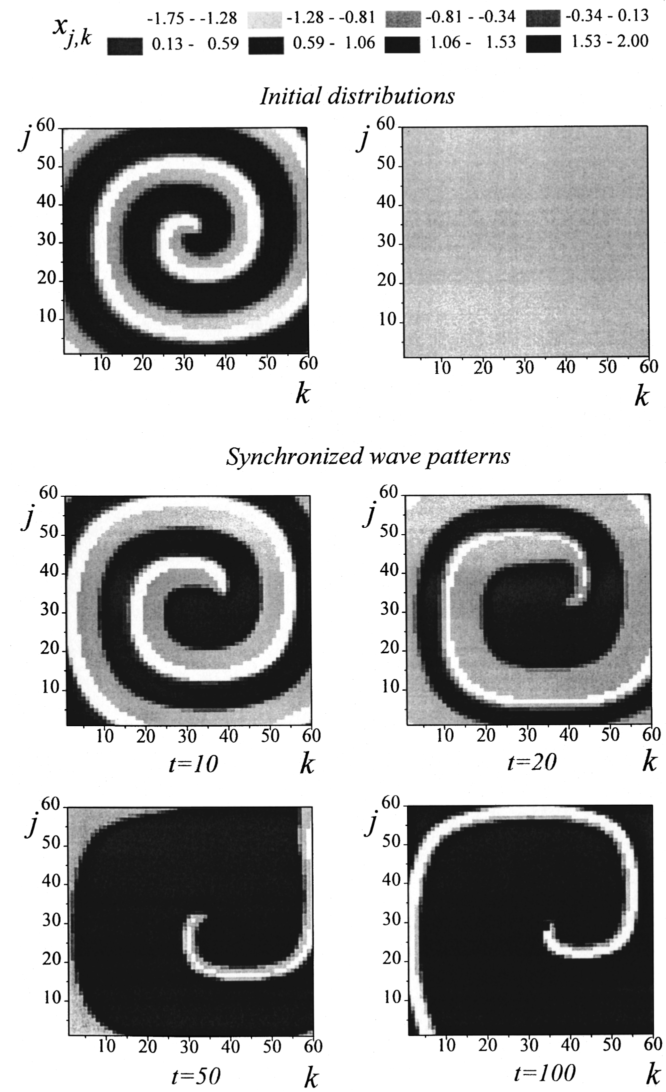

FIG. 9. Time sequence illustrating the transformation of an *oscillatory* spiral into a bright *excitable* spiral wave. Parameter values: $D_1 = D_2 = 0.8$ and $h = 1$.

ing spirals in reaction diffusion systems is of potential interest, in particular in biology and medicine, when studying arrhythmias and fibrillation in heart muscle fibers [18], different reentry phenomena (Ref. [3] and references therein), etc.

Let the first layer of system (1) exhibit a bright *excitable* spiral wave (Fig. 8), while all elements of the second layer are in the rest state, P^-. At a given instant of time we switch on the interlayer coupling. Its strength h is chosen high enough to provide synchronization between the layers. Condition (7) gives an upper value to the coupling coefficient. The actual (numerically obtained) value appears to be smaller, and depends on the kind of synchronizing motions. The interaction provides the entry of the excitation to the initial rest layer, and back to the layer with the rotating spiral. Synchronization occurs rather quickly. The sequence of snapshots given in Fig. 8 shows the development of the spiral wave in the synchronized layers. As in the case of interacting steady patterns, the interlayer interaction leads to the appearance of a wave pattern of new form. It represents a large scale spiral wave of the *oscillating* type (see Sec. III B). Hence the discrete medium, in fact, changes its properties from the excitable to the oscillatory reaction kinetics as the result of the interlattice interaction.

The second example for illustration of wave pattern reentry is the interaction of a large scale *oscillating* spiral in the first layer with the homogeneous rest state in the second. Figure 9 shows the initial distribution and snapshots of the synchronized motions in the layers. The resulting wave pattern represents a bright *excitable* spiral. Thus again there occurs an inversion of the properties of the medium when the elements initially oscillatory [see Figs. 4(b) and 4(c)] return to the excitable behavior [see Figs. 3(b) and 3(c)]. The interaction of bright and oscillating spirals (see Fig. 10) leads to the appearance of a synchronized, ''dark'' spiral wave pattern.

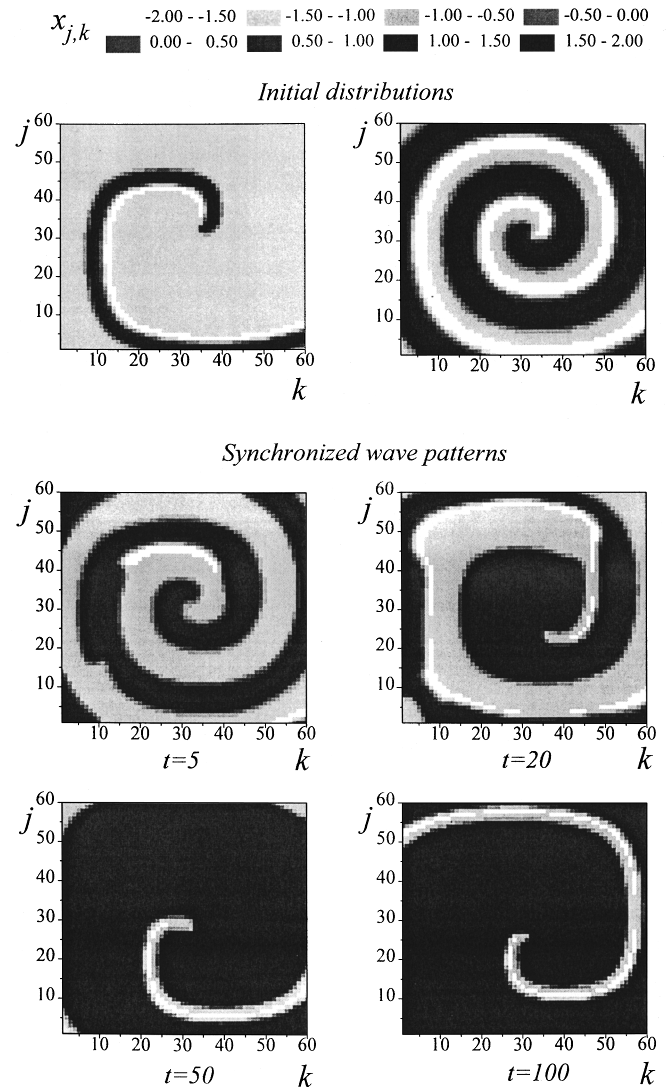

FIG. 10. Time sequence illustrating the reentry of the two different types of spiral waves. Parameter values: $D_1 = D_2 = 0.8$ and $h = 1$.

D. Pattern and wave interaction. Replication of spiral waves

Let us consider the interaction of a spiral wave in one layer with an inhomogeneous steady pattern in the other. This may occur when the intralattice diffusion coefficients have largely different values. To have a steady pattern in a layer, diffusion must be weak enough [Eq. (5)] while traveling waves occur for high enough values of D (Sec. III B). It follows from the results presented in Sec. IV A 2 that the mutual synchronization of two layers with largely different diffusion coefficients requires a very strong interlattice coupling, $h \to \infty$. Figure 11 illustrates the interaction of a disordered steady pattern [Fig. 11(a)] and a dark spiral wave [Fig. 11(b)] for $h = 10$. The layers become almost synchronized, and exhibit the same spiral waves. This situation is similar to the replication of steady patterns in identical layers. But now the lattice initially in the disordered steady state replicates the spiral wave, which is a traveling, rotating pattern. The resulting spirals differ from the original wave. They become ''narrower'' [Figs. 11(c) and 11(d)] as the approximate system describing the synchronized state (Sec. IV A 2) has the diffusion coefficient $D^+ = (D_1 + D_2)/2$. It appears to be smaller than for the initial wave.

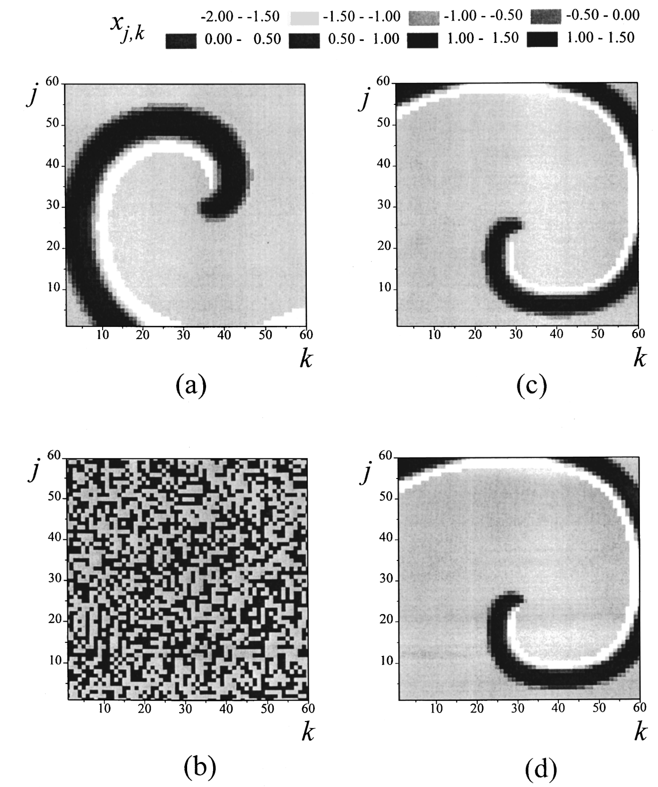

FIG. 11. Replication of a spiral wave pattern. (a) Initial dark *excitable* spiral wave for $D_1 = 2.5$. (b) Steady, spatially disordered state for $D_2 = 0.14$. (c) and (d) Synchronized state ($h = 10$) representing the spiral wave of the same type as the initial stimulus which behaves like the spiral in the lattice with $D \approx 1.3$.

What about the dynamic properties of the layers? Before they interact we have two different layers. The first one with rather strong diffusion describes a medium suitable only for traveling waves, while the second one with weak diffusion could exhibit only steady patterns. The synchronized layers represent, in fact, a single lattice with the diffusion D^+, unable to hold inhomogeneous steady patterns but keeping, however, the capability for spiral waves. Thus, suitably choosing the diffusion coefficients D_1 and D_2 in each layer, we can control the dynamics of the resulting, synchronized discrete medium. It is possible, for example by decreasing D_1 in the first layer, to cause propagation failure of the spiral in the synchronized state when the diffusion coefficient D^+ is not strong enough to support traveling waves.

V. CONCLUSION

In summary, we have shown that in lattices of excitable elements, e.g., modeled by a bistable electronic unit [14]:

(i) Steady patterns are possible for weak enough intralattice diffusion. There is a highly diverse variety (2^{N^2}) of possible pattern profiles including stable, spatially disordered steady patterns.

(ii) When the intralattice diffusion is strong enough, a lattice is capable of supporting excitable spiral waves of ''dark'' and ''bright'' types. They naturally originate from excitable single pulses. Although the units in the lattice do not possess limit cycle behavior, as they have long lasting (metastable) periodiclike stages the lattice can also support oscillatory spiral waves very much like spiral waves in true oscillatory media.

(iii) The interaction between two such lattices with excitable, bistable units leading to their mutual synchronization yields the possibility of replicating steady, regular, and disordered patterns, as well as wavelike behavior including oscillatory and excitable spiral waves typical of reentry processes in 3D reaction-diffusion system.

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APPENDIX: GLOBAL STABILITY OF THE SYNCHRONIZATION MANIFOLD

To prove the global stability of the synchronization manifold M, we consider the Lyapunov function

$$V=\frac{1}{2}\sum_{j,k=1}^{N}\left\{u_{j,k}^2+\alpha v_{j,k}^2+\frac{\alpha}{\beta}w_{j,k}^2\right\},$$

whose derivative with respect to Eq. (6) is

$$\dot{V}=-\sum_{j,k=1}^{N}(P_{j,k}+Q_{j,k}), \tag{A1}$$

with

$$P_{j,k}\equiv(2h+4D)u_{j,k}^2+\alpha u_{j,k}G(u_{j,k},u_{j,k}^{+}) -Du_{j,k}(u_{j-1,k}+u_{j,k-1}+u_{j+1,k}+u_{j,k+1}),$$

$$Q_{j,k}\equiv\alpha(u_{j,k}-v_{j,k})^2+\frac{\alpha\gamma}{\beta}w_{j,k}^2.$$

Then all $Q_{j,k}$'s are positive definite. Let us find the conditions for the function $\Sigma_{j,k=1}^{N}P_{j,k}$ also to be positive definite. The piecewise linear term G satisfies the relation

$$u_{j,k}G(u_{j,k},u_{j,k}^{+})\geqslant -au_{j,k}^2,\quad \forall j,k=1,\ldots,N.$$

Hence

$$\sum_{j,k=1}^{N}P_{j,k}\geqslant\sum_{j,k=1}^{N}\{pu_{j,k}^2-Du_{j,k}(u_{j-1,k}+u_{j,k-1}+u_{j+1,k}+u_{j,k+1})\}\equiv P, \tag{A2}$$

with

$$p=(2h+4D-2\alpha a).$$

Consider the vector $\mathbf{z}=(z_1,z_2,\ldots,z_{N^2})$, where $z_1=u_{11},z_2=u_{12},\ldots,z_{N^2}=u_{NN}$. By using the components of the vector $\mathbf{z}$, the function P takes the quadratic form

$$P=\sum_{i,j=1}^{N^2}a_{ij}z_iz_j\equiv\mathbf{z}^TA\mathbf{z},$$

where $a_{ij}=a_{ji}$, the superscript T denotes the transpose, and $A=\|a_{ij}\|$ is a square symmetric $N^2\times N^2$ matrix. The quadratic form P will be positive definite if the eigenvalues of the symmetric matrix A are positive (see Ref. [30], p. 74). Applying Gershgorin theorem [31] to the matrix A, we find that if $p>4D$ then the union of the Gershgorin disks corresponding to the matrix A is located to the right of the imaginary axis. Therefore, the inequality $h>\alpha a$ ensures the positiveness of all eigenvalues of the matrix A, and the form P is a positive definite quadratic form for conditions (7). Then, using Eq. (A1), it follows that outside the manifold M the inequality $\dot{V}<0$ is satisfied, and $\dot{V}=0$ in the manifold. Thus the synchronization manifold M is globally asymptotically stable.

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